# ON THE SOLUTION AND ELLIPTICITY PROPERTIES OF THE SELF-DUALITY EQUATIONS OF CORRIGAN ET AL. IN EIGHT DIMENSIONS

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**Abstract**

We show that the two sets of self-dual Yang-Mills equations in 8-dimensions proposed in (E.Corrigan, C.Devchand, D.B.Fairlie and J.Nuyts, *Nuclear Physics* **B214**, 452-464, (1983)) form respectively elliptic and overdetermined elliptic systems under the Coulomb gauge condition. In the overdetermined case, the Yang-Mills fields can depend at most on $N$ arbitrary constants, where $N$ is the dimension of the gauge group. We describe a subvariety $\mathcal{P}_8$ of the skew-symmetric $8\times 8$ matrices by an eigenvalue criterion and we show that the solutions of the elliptic equations of Corrigan et al. are among the maximal linear submanifolds of $\mathcal{P}_8$. We propose an eight order action for which the elliptic set is a maximum.

## 1. Introduction.

The self-duality of a 2-form in four dimensions is defined to be the Hodge duality. Self-dual and anti self-dual 2-forms can equivalently be described as eigenvectors of the completely antisymmetric forth rank tensor $\epsilon_{ijkl}$. The latter approach is pursued by Corrigan et al. [Corrigan et al., 1983)], and self-dual 2-forms in $n$ dimensions are defined as eigenvectors of a completely antisymmetric tensor invariant under a subgroup $G$ of $SO(n)$. Then, various linear self-duality equations are obtained by specifying $G$. In this paper we will study two sets of equations in eight dimensions arising from invariance under $SO(7)$. These equations denoted by *Set a* and *Set b* are given in Section 2.

The *Set b* consisting of 21 equations occurs in connection with other definitions of self-duality. The "strongly self-dual" 2-forms defined in [Corrigan et al., 1983)] are characterized by the property that their cofficients $\omega = (\omega_{ij})$ with respect to an orthonormal basis satisfy the equation $\omega\omega^t = \lambda I$, where $\lambda$ is a nonzero constant. It is shown that [Bilge et al., 1996, Bilge, 1995], this definition is equivalent to the self-duality definitions of Grossman [Grossman et al., 1984] and Trautman [Trautman, 1977] and strongly self-dual 2-forms constitute an $n^2 - n + 1$ dimensional submanifold $\mathcal{S}_8 \cup \{0\}$ (see Definition 3.1). In eight dimensions the maximal linear submanifolds of strongly self-dual 2-forms form a six parameter family of seven dimensional spaces, and solutions of *Set b* are among these maximal linear submanifolds [Bilge et al., 1995].

The solutions of *Set a* and *Set b* can be viewed as analogues of self-dual 2-forms in four dimensions from different aspects. The strongly self-dual 2-forms, hence the solutions of *Set b* saturate various topological lower bounds [Bilge et al., 1996, Bilge, 1995], but they form an overdetermined system. In Section 2 we show that the solutions of *Set b* for an $N$ dimensional gauge group, depend exactly on $N$ arbitrary constants, provided that the system is consistent. Thus the *Set b* lacks the rich structure of the self-duality equations in four dimensions. On the other hand, the solutions of *Set a* do not saturate the topological lower bounds obtained in [Bilge et al., 1996, Bilge, 1995], but these equations form an elliptic system under the Coulomb gauge condition.

In Section 3, we give an eigenvalue criterion to define a subvariety $\mathcal{P}_8$ of $8 \times 8$ skew-symmetric matrices and we show that it contains the solutions of *Set a* as a maximal linear submanifold. We give an eight order action whose extrema are achived on $\mathcal{P}_8$.

## 2. The self-duality equations of Corrigan et al.

We will study the self-duality equations (3.39) and (3.40) in [Corrigan et al., 1983)], that describe a scalar field $F$ which is an eigenvector of a fourth rank tensor $T$ invariant under $SO(7)$. We present below the two sets of equations corresponding to the eigenvalues 1 and $-3$ of $T$. The first set corresponding to the eigenvalue 1 is given below. In the following $\omega$ will denote a 2-form, and $\omega_{ij}$ will be its components with respect to an orthonormal basis.

*Set a:*

$$\begin{aligned}
\omega_{12} + \omega_{34} + \omega_{56} + \omega_{78} &= 0, \\
\omega_{13} - \omega_{24} + \omega_{57} - \omega_{68} &= 0, \\
\omega_{14} + \omega_{23} - \omega_{67} - \omega_{58} &= 0, \\
\omega_{15} - \omega_{26} - \omega_{37} + \omega_{48} &= 0, \\
\omega_{16} + \omega_{25} + \omega_{38} + \omega_{47} &= 0, \\
\omega_{17} - \omega_{28} + \omega_{35} - \omega_{46} &= 0, \\
\omega_{18} + \omega_{27} - \omega_{36} - \omega_{45} &= 0.
\end{aligned} \tag{2.1}$$

The second set is obtained by equating the terms in each row, i.e.,

*Set b:*

$$\omega_{12} = \omega_{34} = \omega_{56} = \omega_{78}, \quad \ldots. \tag{2.2}$$

Note that the *Set a* is the orthogonal complement of *Set b* with respect to the standart inner product on matrices, $\langle A, B\rangle = tr\ AB^t$.

### 2.1. The number of free parameters in the solution of *Set b*.

Let $F$ be the curvature 2-form, $F = \sum_{a,b} F_{ab}E_{ab}$ where the $E_{ab}$'s are basis vectors for the Lie algebra of the gauge group. Assume that each 2-form $F_{ab}$ satisfies the equations in *Set b*, or more generally belongs to any linear submanifold of $\mathcal{S}_8 \cup \{0\}$. As these equations are overdetermined, there may not be any solutions. We recall that a topologically nontrivial solution (i.e. where $F$ is not an exact form) is given by Grossman et. al. [Grossman et al., 1984]. Here we will show that, for an $N$ dimensional gauge group, if the field equations are consistent, then the solution depends at most on $N$ arbitrary constants.

The *Set b* represents 21 equations for the 8 components of the connection 1-form. In addition, if we impose the Coulomb gauge condition, for each $F_{ab}$ we have a system of 22 equations for 8 unknowns. However the integrability conditions of the equations for the connection 1-form become quickly very cumbersome. Thus, instead of looking at the compatibility of the differential equations for the connection, we look at the Bianchi identities, which are viewed as first order differential equation for the curvature, i.e.

$$dF_{ab} = A_{ac}F_{cb} - F_{ac}A_{cb}. \tag{2.3}$$

If each 2-form $F_{ab}$ satisfies the equations in *Set b* or more generally belongs to a linear submanifold of $\mathcal{S}_8 \cup \{0\}$, it can be written as $F_{ab} = \sum_{i=1}^{7} F^i_{ab} h'_i$, with respect to some basis $\{h'_i\}$ (one set is actually given by Eq.(2.9)). Then

$$dF_{ab} = \sum_{i=1}^{7}\sum_{j=1}^{8} \partial_j F^i_{ab} dx^j h'_i. \tag{2.4}$$

Thus the Bianchi identities, which are 3-form equations consist of sets of 56 algebraic equations for the 56 partial derivatives $\partial_j F^i_{ab}$, for each pair of indices $(ab)$. It is checked that this system is nondegenerate, therefore if the gauge group is abelian, then the Bianchi identities reduce to homogeneous equations and the $F_{ab}$'s are constants. In the nonabelian case, the Bianchi identities form an inhomogeneous system, from which all partial derivatives of the $F_{ab}$'s are determined. Therefore, if the gauge field equations for the connection are consistent, then the resulting curvature 2-forms $F_{ab}$'s can depend at most on one arbitarary constant for each pair of indices $(ab)$. Thus we have

**Proposition 2.1.** *Let $F = dA - A \wedge A$ where $A$ belongs to an $N$ dimensional Lie algebra, and $F_{ab} =$ satisfy the equations in Set b. Then, if the system is compatible, $F$ can depend at most on $N$ arbitrary constants.*

### 2.2. Ellipticity properties of *Set a* and *Set b*.

Recall that $F = dA - A \wedge A$, and the characteristic determinant [John, 1982] of the field equations are obtained using the linear part of this equation, i.e $F \sim dA$. The Coulomb gauge condition is

$$\sum_{i}^{n} \partial_i A^i = 0. \tag{2.5}$$

The characteristic determinant of the *Set a* together with the Coulomb gauge condition is obtained and we have,

**Proposition 2.2.** *The characteristic determinant of the Set a, together with the Coulomb gauge condition is*

$$(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2 + \xi_8^2)^4 \tag{2.6}$$

*hence the system is elliptic.*

A system of elliptic equations should have as many equations as unknowns. The requirement of ellipticity is the injectivity and the surjectivity of the symbol. If the symbol is injective (but not surjective), then the system is called *overdetermined elliptic*. The injectivity of the symbol leads to certain inequalities in terms of various Sobolev norms [Donaldson and Kronheimer, 1990]. On the other hand the surjectivity of the symbol guarantees the solvability of the system. Thus if a system is overdetermined elliptic, one can still use standart results from elliptic theory, provided that the existence of solutions to the overdetermined system are guaranteed.

The *Set b* together with the Coulomb gauge condition is an overdetermined system. The subsystem consisting of the equations

$$\begin{array}{llll} \omega_{12} = \omega_{34}, & \omega_{13} = -\omega_{24}, & \omega_{14} = \omega_{23}, & \omega_{15} = -\omega_{26}, \\ \omega_{16} = \omega_{25}, & \omega_{17} = -\omega_{28}, & \omega_{18} = \omega_{27} & \end{array} \tag{2.7}$$

together with the Coulomb gauge condition form an elliptic system.

**Proposition 2.3.** *The characteristic determinant of the Eqs.(2.7) together with the Coulomb gauge condition is*

$$(\xi_1^2 + \xi_2^2)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 + \xi_7^2 + \xi_8^2) \tag{2.8}$$

*hence the Set b is overdetermined elliptic.*

We note that the rank of the characteristic system of the *Set b* consisting of 21 equations is 7, hence the Coulomb gauge condition is needed in order to obtain a full rank subsystem.

**Remark 2.4.** The characteristic matrix $A$ of the *Set a* satisfies the equation $AA^t = kI$ where $t$ denotes the transpose, $I$ is the identity matrix and $k = \sum_{i=1}^{8} \xi_i^2$. The first row of the characteristic determinant, arising from the Coulomb gauge condition, is the radial vector, hence the remaining seven rows represent tangent vector fields to $S^7$. Since $S^3$ and $S^7$ are the only parallelizable spheres, the equations in *Set a* are unique analogues of the self-duality equations in four dimensions, as already noted in [Corrigan et al., 1983)].

### 2.3 An alternative derivation of the *Set a* and *Set b*.

We recall that squares of strongly-self dual 2-forms are self-dual in the Hodge sense [Bilge et al., 1996)] and the maximal linear subspaces of strongly self-dual 2-forms are a six parameter family of 7 dimensional spaces. In this section we will obtain analogues of Eqs.(2.2) that will be used in Section 3. Similar equations are also obtained in [Bilge et al., 1995].

We fix a nondegenerate 2-form $h_1' = e_{12} + \alpha e_{34} + \beta e_{56} + \gamma e_{78}$, and we consider the 2-forms $h_j' = e_{1(j+1)} + \kappa_j'$, for $j = 2, \dots 7$, such that the $\kappa_j'$'s do not involve $e_1$ and $e_{j+1}$. The requirement that $(h_1' + h_j')^2$ be self-dual gives linear equations for the components of the $h_j'$'s. Once these equations are solved, the non-linear equations obtained from the self-duality of $(h_i' + h_j')^2$ for $i \neq 1$ can be solved easily and we obtain the following result.

**Proposition 2.6.** *Let $h_1' = e_{12} + \alpha e_{34} + \beta e_{56} + \gamma e_{78}$, and $h_j'$, $j = 2, \dots 7$ be of the form $h_j' = e_{1(j+1)} + \kappa_j'$, such that $\langle e_1, \kappa_j' \rangle = \langle e_{j+1}, \kappa_j' \rangle = 0$. If the 4-forms $(h_i' + h_j')^2$ are self dual for all $i, j$ then the $h_i'$'s are*

$$
\begin{aligned}
h_1' &= e_{12} + \beta\gamma e_{34} + \beta e_{56} + \gamma e_{78} \\
h_2' &= e_{13} - \beta\gamma e_{24} + \beta c'\ e_{57} - \beta c\ e_{58} - \gamma c\ e_{67} - \gamma c'\ e_{68} \\
h_3' &= e_{14} + \beta\gamma e_{23} - c\ e_{57} - c'\ e_{58} - \beta\gamma c'\ e_{67} + \beta\gamma c\ e_{68} \\
h_4' &= e_{15} - \beta e_{26} - \beta c'\ e_{37} + \beta c\ e_{38} + c\ e_{47} + c'\ e_{48} \\
h_5' &= e_{16} + \beta e_{25} + \gamma c\ e_{37} + \gamma c'\ e_{38} + \beta\gamma c'\ e_{47} - \beta\gamma c\ e_{48} \\
h_6' &= e_{17} - \gamma e_{28} + \beta c'\ e_{35} - \gamma c\ e_{36} - c\ e_{45} - \beta\gamma c'\ e_{46} \\
h_7' &= e_{18} + \gamma e_{27} - \beta c\ e_{35} - \gamma c'\ e_{36} - c'\ e_{45} + \beta\gamma c\ e_{46}
\end{aligned} \tag{2.9}
$$

*where $\beta^2 = \gamma^2 = c^2 + c'^2 = 1$.*

Thus depending on the possible choices for $\beta$ and $\gamma$ we have four sets of seven equations parametrized by $c$ and $c'$. We denote these forms by $h_i'$, $k_i'$, $m_i'$ and $n_i'$ corresponding respectively to the cases $(\beta = 1, \gamma = 1)$, $(\beta = 1, \gamma = -1)$, $(\beta = -1, \gamma = 1)$ and $(\beta = -1, \gamma = -1)$.

The set consisting of the 28 forms thus obtained is however linearly dependent for any $c$ and $c'$. To retain similarity with Eq.(2.2) we set $c' = 1$ and $c = 0$, and we obtain the following linear submanifolds of $\mathcal{S}_8 \cup \{0\}$.

$$
\begin{aligned}
B^{++} =& \text{span}\{h_1', h_2', h_3', h_4', h_5', h_6', h_7'\} \\
B^{+-} =& \text{span}\{k_1', k_2', k_3', h_4', k_5', k_6', k_7'\},
\end{aligned}
$$

$$
\begin{aligned}
B^{-+} =&\mathrm{span}\{m_1', m_2', k_3', m_4', m_5', m_6', h_7'\},\\
B^{--} =&\mathrm{span}\{n_1', n_2', h_3', m_4', n_5', n_6', k_7'\}.
\end{aligned} \tag{2.10}
$$

A basis for 2-forms on $R^8$ can be obtained by adding

$$
\begin{aligned}
p_1' =&e_{14} - e_{23} + e_{58} - e_{67},\\
p_2' =&e_{14} + e_{23} + e_{58} + e_{67},\\
p_3' =&e_{15} + e_{26} - e_{37} - e_{48},\\
p_4' =&e_{15} - e_{26} + e_{37} - e_{48},\\
p_5' =&e_{18} + e_{27} + e_{36} + e_{45},\\
p_6' =&e_{18} - e_{27} - e_{36} + e_{45},
\end{aligned} \tag{2.11}
$$

to the 2-forms in (2.10).

The analogues of the equations in *Set b* are obtained by resticting $\omega$ to the subspaces in (2.10). Similarly the analogues of *Set a* are obtained by taking orthogonal complements. The coefficients of $\omega$ with respect to the basis consisting of the $h_i'$'s, $k_i'$'s, $m_i'$'s, $n_i'$'s and $p_i'$'s will be denoted by the same symbols without prime.

## 3. An eigenvalue characterization of the *Set a* and an action density.

We recall the following definition given in [Bilge et al., 1996)].

**Definition 3.1.** Let $\omega$ be a 2-form in $2n$ dimensions, with components $\omega_{ij}$ with respect to an orthonormal basis. The 2-form $\omega$ is called *strongly self-dual* if the the absolute values of the eigenvalues of the matrix $\omega_{ij}$ are equal. The non-zero strongly self-dual 2-forms belong to a 13 dimensional submanifold $\mathcal{S}_8$, and the solutions of the *Set b* are among the maximal linear submanifolds of $\mathcal{S}_8 \cup \{0\}$ [Bilge et al., 1995].

We will define below a subvariety $\mathcal{P}_8$ which contains the solutions of *Set a* as a maximal linear submanifold.

Let the eigenvalues of the matrix $\omega_{ij}$ be $\pm i\lambda_k$, $k = 1, \ldots, 4$, and define $q_j$ to be the $j$'th elementary symmetric function of the $\lambda_k^2$'s. Then

$$
(\omega, \omega) =s_2 = 4q_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2,
$$

$$
\begin{aligned}
\tfrac{1}{2^2}(\omega^2,\omega^2) =& s_4 = 6q_2 = \lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_1^2\lambda_4^2+\lambda_2^2\lambda_3^2+\lambda_2^2\lambda_4^2+\lambda_3^2\lambda_4^2,\\
\tfrac{1}{6^2}(\omega^3,\omega^3) =& s_6 = 4q_3 = \lambda_1^2\lambda_2^2\lambda_3^2+\lambda_1^2\lambda_2^2\lambda_4^2+\lambda_1^2\lambda_3^2\lambda_4^2+\lambda_2^2\lambda_3^2\lambda_4^2,\\
\tfrac{1}{24^2}(\omega^4,\omega^4) =& s_8 = q_4 = \lambda_1^2\lambda_2^2\lambda_3^2\lambda_4^2.
\end{aligned}
\tag{3.1}
$$

We have the inequalities

$$
q_1^2 \geq q_2 \geq \sqrt{q_4}, \tag{3.2}
$$

the equalities being saturated iff all the eigenvalues are equal [10], i.e. for the strongly self-dual forms. This corresponds to the case where the quantities

$$
\begin{aligned}
A &= (\omega,\omega)^2 - \tfrac{2}{3}(\omega^2,\omega^2),\\
B &= (\omega^2,\omega^2) - (\omega^4,\omega^4)^{1/2}
\end{aligned}
\tag{3.3}
$$

vanish. The proposition 3.2 below implies that the quantity

$$
\Phi = A + \frac{1}{3}B = (\omega,\omega)^2 - \frac{1}{3}(\omega^2,\omega^2) - \frac{1}{3}(\omega^4,\omega^4)^{1/2} \tag{3.4}
$$

is a measure of the power of the anti self-dual part of $\omega$.

**Proposition 3.2.** *Let $(\omega^{2+},\omega^{2+}) \geq (\omega^{2-},\omega^{2-})$, and $\Phi = (\omega,\omega)^2 - \alpha(\omega^{2+},\omega^{2+})$, where $\omega^{2\pm}$ denote the self-dual and anti self-dual parts of $\omega^2$. Then,* max *$\alpha$ such that $\Phi$ is non-negative for all $\omega$ is $\alpha = \frac{3}{2}$.*

*Proof.* If $(\omega^{2+},\omega^{2+}) \geq (\omega^{2-},\omega^{2-})$, then $(\omega^4,\omega^4)^{1/2} = *\omega^4 = (\omega^{2+},\omega^{2+}) - (\omega^{2-},\omega^{2-})$. From the inequalities (3.2), it can be seen that $\alpha \leq 3/2$. On the other hand the equality is attaigned for $\omega \in \mathcal{S}_8$, hence $\alpha = 3/2$. e.o.p.

It is an elementary fact that the product $\frac{1}{3}AB$, under the constraint $A + \frac{1}{3}B$ =const. is maximized for $\Psi = A - \frac{1}{3}B = 0$, and minimized for $A = 0$ or $B = 0$. The condition $A - \frac{1}{3}B = 0$ gives

$$
\Psi = (\omega,\omega)^2 - (\omega^2,\omega^2) + \frac{1}{3}(\omega^4,\omega^4)^{1/2} = 0. \tag{3.5}
$$

Thus we have

**Proposition 3.3.** *Let $\Phi = (\omega,\omega)^2 - \frac{1}{3}(\omega^2,\omega^2) - \frac{1}{3}(\omega^4,\omega^4)^{1/2}$ be fixed and $(\omega,\omega)^2 - \frac{2}{3}(\omega^2,\omega^2)$ be nonzero. Then the quantity $(\omega,\omega)^2 - \frac{2}{3}(\omega^2,\omega^2)(\omega^2,\omega^2) - (\omega^4,\omega^4)^{1/2}$ is maximal for $\Psi = (\omega,\omega)^2 - (\omega^2,\omega^2) + \frac{1}{3}(\omega^4,\omega^4)^{1/2} = 0$.*

The expression of $\Psi$ in terms of the $\omega_{ij}$'s is very complicated. However it reduces to a relatively simple form under a change of parameters. If we reparametrize the eigenvalues as

$$\begin{aligned}\epsilon_1 =& (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4),\\ \epsilon_2 =& (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4),\\ \epsilon_3 =& (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4),\\ \epsilon_4 =& (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4),\end{aligned} \tag{3.6}$$

then $\Psi$ reduces to

$$\Psi = (\omega, \omega)^2 - (\omega^2, \omega^2) + \frac{1}{3}(\omega^4, \omega^4)^{1/2} = \epsilon_1\epsilon_2\epsilon_3\epsilon_4. \tag{3.7}$$

We note that we could obtain a similar decomposition for $(\omega, \omega)^2 - (\omega^2, \omega^2) - \frac{1}{3}(\omega^4, \omega^4)^{1/2}$, if we defined the $e_i$'s with an odd number of minus signs.

The equality of the $\lambda_i$'s corresponds to the case where any three of the $\epsilon_i$'s are zero. The appropriate nonlinear set containing solutions of *Set a* is thus the set where only one of the $\epsilon_i$'s is zero. The explict expression of $\Psi$ when $\omega$ is written with respect to the basis given in (2.10) and (2.11) is given below.

$$\begin{aligned}\Psi = h_1[&k_1(m_1n_1 + m_4p_3 + m_5n_5) + k_2(n_1m_2 - m_4p_6 + m_5n_6) + k_3(n_1p_1 - p_3n_6 - n_5p_6)\\ &+k_6(m_1n_6 + m_4p_1 - n_5m_2) + k_7(m_1p_6 + p_3m_2 + m_5p_1)\\ +h_2[&k_1(m_1n_2 + m_4p_5 + n_5m_6) + k_2(m_4p_4 + m_2n_2 + n_6m_6) + k_3(n_5p_4 - n_6p_5 + p_1n_2)\\ &+k_5(-m_1n_6 - m_4p_1 + n_5m_2) + k_7(-m_1p_4 + m_2p_5 + p_1m_6)\\ +h_3[&k_1(m_1p_2 + p_3m_6 - m_5p_5) + k_2(-m_5p_4 + m_2p_2 - p_6m_6) + k_3(p_3p_4 + p_6p_5 + p_1p_2)\\ &+k_5(m_1p_6 + p_3m_2 + m_5p_1) + k_6(-m_1p_4 + m_2p_5 + p_1m_6)\\ +h_4[&m_1(n_1p_4 + p_6n_2 + n_6p_2) + m_2(-n_1p_5 + p_3n_2 - n_5p_2) + m_4(p_3p_4 + p_6p_5 + p_1p_2)\\ &+m_5(n_5p_4 - n_6p_5 + p_1n_2) + m_6(-n_1p_1 + p_3n_6 + n_5p_6)\\ +h_5[&k_2(-n_1m_6 + m_4p_2 + m_5n_2) + k_3(n_1p_5 - p_3n_2 + n_5p_2) + k_5(m_1n_1 + m_4p_3 + m_5n_5)\\ &+k_6(m_1n_2 + m_4p_5 + n_5m_6) + k_7(-m_1p_2 - p_3m_6 + m_5p_5)\\ +h_6[&k_1(n_1m_6 - m_4p_2 - m_5n_2) + k_3(n_1p_4 + p_6n_2 + n_6p_2) + k_5(n_1m_2 - m_4p_6 + m_5n_6)\\ &+k_6(m_4p_4 + m_2n_2 + n_6m_6) + k_7(m_5p_4 - m_2p_2 + p_6m_6)\\ +h_7[&k_1(n_1p_5 - p_3n_2 + n_5p_2) + k_2(n_1p_4 + p_6n_2 + n_6p_2) + k_5(-n_1p_1 + p_3n_6 + n_5p_6)\\ &+k_6(-n_5p_4 + n_6p_5 - p_1n_2) + k_7(p_3p_4 + p_6p_5 + p_1p_2)\end{aligned} \tag{3.8}$$

From (3.8), it can be seen that $\Psi = 0$ both on the *Set a* where all all $h_i$'s are zero, and on the *Set b* where all the parameters except the $h_i$'s are zero. Actually $\Psi$ vanishes on the complement of each of the subspaces in (2.10), which are 21 dimensional linear submanifolds of $\mathcal{P}_8$. By assigning arbitrary values to the remaining parameters, it can be seen that these 21 dimensional submanifolds are maximal. Hence solutions of *Set a* (and their analogues) are among the maximal linear submanifolds of $\mathcal{P}_8$.

**Acknowledgements.** This work is partially supported by the Turkish Scientific and Technological Research Council, TUBITAK.

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